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Bi-normal semigroups

by

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1. Let S be a semigroup. Following the notation and terminology of A. H. Clifford and G. B. Preston [1] we shall say that S is a semilattice of groups if it is the set-theoretical union of a family $\{S_\alpha: \alpha \in M\}$ of mutually disjoint subgroups S_α such that, for every $\alpha, \beta \in M$, the product $S_\alpha S_\beta$ and $S_\beta S_\alpha$ are both contained in the same subgroup $S_\gamma (\gamma \in M)$.

In this paper we shall introduce a weakened notion of commutative semigroups, which is called bi-normal semigroups, and characterize semigroups that are semilattices of groups in terms of bi-normality.

2. A subsemigroup B of a semigroup S is called a *bi-ideal* of S if $BSB \subseteqq B$. As is easily seen, the set $B(S)$ of all bi-ideals of a semigroup S is also a semigroup under the multiplication of subsets. A bi-ideal B of a semigroup S is called *two-sided pure*, or simply *T-pure*, if

$$B \cap xSy = xBy$$

holds for all elements x and y of S, and S is called *T^*-pure* if every bi-ideal of it is T-pure (see [2]). A semigroup S is called *regular* if, for each element a of S, there exists an element x in S such that $a = axa$. A semigroup S is called *bi-normal* if $xSy = ySx$ holds for all elements x and y of S.

Now we list some results which will be needed in the following discussions.

LEMMA 2.1 ([2] Theorem 11). *For a T^*-pure semigroup S the following conditions are equivalent.*

(1) *S is regular.*

(2) *$B(S)$ is idempotent.*

LEMMA 2.2 ([2] Lemma 4). *Let A be any T-pure bi-ideal of a semigroup S. Then*

$$A \cap XSY = XAY .$$

holds for all bi-ideals X and Y of S.

LEMMA 2.3 ([3] Theorem 9). *For a semigroup S the following* *conditions are equivalent.*

(1) *S is a semilattice of groups.*

(2) *S is regular and T^*-pure.*

3. We denote by $[x]_R$, $[x]_L$ and $[x]_B$ the principal right ideal, left ideal and bi-ideal of a semigroup S generated by x in S. First we have the following.

THEOREM 3.1. *For a semigroup S the following conditions are equivalent.*

(1) *S is bi-normal.*

(2) *$XSY = YSY$ for all non-empty subsets X and Y of S.*

(3) *$XSY = YSX$ for all $X, Y \in B(S)$.*

(4) *$[x]_B S[y]_B = [y]_B S[x]_B$ for all $x, y \in S$.*

(5) *$[x]_R S[y]_L = [y]_R S[x]_L$ for all $x, y \in S$.*

Proof. Assume that S is bi-normal. Let X and Y be any non-empty subsets of S and xsy $(x \in X, s \in S, y \in Y)$ be any element of XSY. Then we have

$$xsy \in xSy = ySx \subseteqq YSX$$

and so we have

$$XSY \subseteqq YSX .$$

Since it can be similarly proved that the converse inclusion holds, we obtain that

$$XSY = YSX$$

holds for all non-empty subsets X and Y of S, and that (1) implies (2). It is clear that (2) implies (3), and that (3) implies (4) and (5). Assume that (5) holds. Let x and y be any elements of S. Then we have

$$\begin{aligned} xSy &\subseteqq [x]_R S[y]_L = [y]_R S[x]_L = (y \cup yS)S(x \cup Sx) \\ &= ySx \cup yS(Sx) \cup (yS)Sx \cup (yS)S(Sx) \\ &= ySx \cup y(SS)x \cup y(SS)x \cup y(SSS)x \\ &\subseteqq ySx . \end{aligned}$$

Similarly we have

$$ySx \subseteqq xSy$$

for all $x, y \in S$. Thus we obtain that

$$xSy = ySx$$

for all $x, y \in S$, and that (5) implies (1). Similarly it can be proved that (4) implies (1). This completes the proof of the theorem.

THEOREM 3.2. *For a semigroup S the following conditions are equivalent.*

(1) *S is bi-normal.*

(2) *$B(S)$ is bi-normal.*

Proof. Assume that S is bi-normal. Let X, Y and Z be any bi-ideals of S and x and y respectively any elements of X and Y. Then we have

$$xZy \subseteq xSy = ySx \subseteq YSX \subseteq YB(S)X$$

and so we have

$$XB(S)Y \subseteq YB(S)X .$$

It can be similarly proved that the converse inclusion holds. Thus we obtain that

$$XB(S)Y = YB(S)X$$

holds for all $X, Y \in B(S)$ and that (1) implies (2).

Conversely assume that $B(S)$ is bi-normal. Let x and y be any elements of S. Then for some $A \in B(S)$ we have

$$xSy = [x]_B S[y]_B = [y]_B A[x]_B \subseteq [y]_B S[x]_B = ySx .$$

Similarly we have

$$ySx \subseteq xSy$$

for all $x, y \in S$. Thus we obtain that

$$xSy = ySx$$

for all $x, y \in S$, and that (2) implies (1). This completes the proof of the theorem.

It is clear that any commutative semigroup is bi-normal. For idempotent semigroups we have the following.

THEOREM 3.3. *For an idempotent semigroup S the following conditions are equivalent.*

(1) *S is bi-normal.*

(2) *S is commutative.*

Proof. It suffices to prove that (1) implies (2). Assume that S is bi-normal. Let x and y be any elements of S. Then for some $z \in S$ we have

$$xy = xxy = yzx .$$

Then we have

$$yx = (yx)(yx) = y(xy)x = y(yzx)x = (yy)z(xx) = yzx = xy .$$

Therefore, S is commutative. This completes the proof.

4. A semigroup S is called completely regular if, for each element x of S, there exists an element x' in S such that

$$x = xx'x \quad \text{and} \quad xx' = x'x .$$

Such a semigroup is characterized as follows: A semigroup S is completely regular if and only if $x \in x^2Sx^2$ for all $x \in S$, (cf. [5] p. 426).

THEOREM 4.1. *For a bi-normal semigroup S the following conditions are equivalent.*

(1) *S is regular.*

(2) *S is completely regular.*

Proof. It is clear that (2) implies (1). Assume that S is regular. Let x be any element of S. Then by Theorem 3.1 we have

$$x \in xSx \subseteq xSxSxSx = x\{(Sx)S(xS)\}x = x\{(xS)S(Sx)\}x = x^2S^3x^2 \subseteq x^2Sx^2 .$$

Thus S is completely regular, and (1) implies (2).

THEOREM 4.2. *For a completely regular semigroup S the following conditions are equivalent.*

(1) *S is bi-normal.*

(2) *$eSf = fSe$ for all idempotent elements e and f of S.*

Proof. It is clear that (1) implies (2). Assume that (2) holds. Let x and y be any elements of S. Then, since S is completely regular, there exist elements x' and y' in S such that

$$x = xx'x, \ xx' = x'x, \ y = yy'y \quad \text{and} \quad yy' = y'y .$$

Since xx' and $y'y$ are idempotent elements of S, we have

$$\begin{aligned} xSy &= (xx'x)S(yy'y) = (xx')(xSy)(y'y) \\ &\subseteq (xx')S(y'y) = (y'y)S(xx') \\ &= (yy')S(x'x) = y(y'Sx')x \subseteq ySx . \end{aligned}$$

It can be similarly proved that the converse inclusion holds. Thus we obtain that

$$xSy = ySx$$

for all $x, y \in S$, and that (2) implies (1).

THEOREM 4.3. *For a regular semigroup S the following conditions are equivalent.*

(1) *S is bi-normal.*

(2) *S is T^*-pure.*

Proof. First assume that S is bi-normal. Let A be any bi-ideal of S and x and y any elements of S. Let $a = xsy$ ($a \in A$, $s \in S$) be any element of $A \cap xSy$. Since S is regular, there exist elements x', y' and a' in S such that

$$x = xx'x\ ,\quad y = yy'y \quad \textit{and} \quad a = aa'a\ .$$

Then we have

$$\begin{aligned} a &= xsy = (xx'x)s(yy'y) = (xx')(xsy)(y'y) \\ &= (xx')a(y'y) = (xx')(aa'a)(y'y) \\ &= x\{(x'a)a'(ay')\}y \in x\{(x'a)S(ay')\}y \\ &= x\{(ay')S(x'a)\}y = x\{a(y'Sx')a\}y \\ &\subseteqq x(ASA)y \subseteqq xAy \end{aligned}$$

and so we have

$$A \cap xSy \subseteqq xAy\ .$$

Let xay $(a \in A)$ be any element of xAy. Then we have for some $a' \in S$ such that $a = aa'a$,

$$\begin{aligned} xay &= x(aa'a)y = (xa)a'(ay) \in (xa)S(ay) \\ &= (ay)S(xa) = a(ySx)a \subseteqq ASA \subseteqq A \end{aligned}$$

and so we have

$$xAy \subseteqq A\ .$$

Since

$$xAy \subseteqq xSy$$

for all x, $y \in S$, we have

$$xAy \subseteqq A \cap xSy$$

for all x, $y \in S$. Thus we obtain that

$$A \cap xSy = xAy$$

for all x, $y \in S$, that is, A is T-pure. Thus we obtain that S is T^*-pure, and that (1) implies (2).

Conversely, assume that S is T^*-pure. Let X and Y be any bi-ideals of S. Since S is regular, it follows from Lemma 2.1 that $B(S)$ is idempotent. Then it follows from Lemma 2.2 that

$$XY = (XY)^2 = X(YX)Y = YX \cap XSY \subseteqq YX\ .$$

Similarly, we have

$$YX \subseteqq XY\ .$$

Thus we obtain that

$$XY = YX\ .$$

Since $B(S)$ is idempotent, it follows from Theorem 3.3 that $B(S)$ is bi-normal. Then it follows from Theorem 3.2 that S is bi-normal. Therefore we obtain that (2) implies (1). This completes the proof of the theorem.

5. Finally we characterize semigroups that are semilattices of groups. For another characterizatins, see S. Lajos [4].

THEOREM 5.1. *For a semigroup S the following conditions are equivalent.*

(1) *S is a semilattice of groups.*

(2) *S is regular and bi-normal.*

(3) *S is completely regular and bi-normal.*

(4) *S is completely regular and $eSf=fSe$ for all idempotent elements e and f of S.*

Proof. It follows from Lemma 2.3 and Theorem 4.3 that (1) and (2) are equivalent. The equivalence of (2) and (3) is due to Theorem 4.1, and of (3) and (4) is due to Theorem 4.2. This completes the proof of the theorem.

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